

Long-wave and short-wave asymptotics in nonlinear dispersive systems

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In this paper we study the interplay between short- and long-space scales in the context of conservative dispersive systems. We consider model systems in (1+1) dimensions that admit both long- and short-wavelength solutions in the linear regime. A nonlinear analysis of these systems is constructed, making use of multiscale expansions. We show that the equations governing the lowest order involve only short-wave properties and that the long-wave effects to leading order are determined by a secularity elimination procedure. [S1063-651X(99)02708-7]

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I. INTRODUCTION

In this paper we will consider conservative dispersive systems in (1+1) dimensions which exhibit the property of admitting, in the linear regime, *both* long- and short-wave solutions. Admissibility means in our context that, when the nonlinear terms of a given system of partial differential equations are dropped, and when we look to the dispersion relation of the resulting system by seeking solutions of the form

$$u(x,t) \propto \exp[i[kx - \omega(k)t]], \quad (1)$$

we will find that $\omega(k)$ is bounded for $k \rightarrow 0$ as well as for $k \rightarrow \infty$. More specifically, we will suppose that

$$\lim_{k \rightarrow 0} \omega(k)/k = A_0 + A_2 k^2 + A_4 k^4 + \dots, \quad (2)$$

$$\lim_{k \rightarrow \infty} \omega(k)/k = B_0 + \frac{B_2}{k^2} + \frac{B_4}{k^4} + \dots \quad (3)$$

Thus, for this kind of system, long and short waves evolve on the same time scale. Before proceeding, let us discuss the scope of this assumption. First of all, we have introduced constants A_0 and B_0 in the above relations. We could have, obviously two different constants. Such a difference between constants could be later scaled away. Thus the results we are going to derive are valid for systems whose dispersion relations are bounded in the above limits. A prominent system satisfying these assumptions is the propagation of electromagnetic waves in saturated ferrites [1]. Other examples come from hydrodynamics, such as those described by the Camassa-Holm equation [2] or the (nonintegrable) Boussinesq system [3]. The purpose of this paper is to establish a nonlinear theory for these systems. In particular, does nonlinearity affect differently short- and long-space scales? To tackle the problem we will resort to perturbative expansions. By means of multiple-scale perturbative methods [4] we can naturally deal with different space scales, by introducing scaled variables representing short- and long-space scales. As to time variables, only long-time variables will be needed, a consequence of Eqs. (2), (3). Furthermore, we need to fix the relative strength of the nonlinearity by making

scaling hypothesis about the typical amplitude of the fields. This will be discussed in Sec. II, where we also derive results concerning different nonlinear regimes. We will find that the lowest order equation involves only short-wave variables, leaving the long-wave dynamics undetermined. However, when going to the next order, secularities appear, demanding thus a secularity elimination procedure, which can be achieved if particular solutions for the lowest order are considered. As a result of this procedure, we will find that the dependence on the long-wave variable is fixed in the lowest-order solution of the perturbative expansion. In Sec. III we summarize our results.

In order to have a clear understanding of the perturbative expansions we will settle our treatment on an explicit equation satisfying the above hypothesis (2), (3). We will consider that our unperturbed system is described by the Benjamin-Bona-Mahony (BBM) equation [5]

$$u_t + u_x - 6uu_x - u_{xxt} = 0. \quad (4)$$

The purpose of considering this equation is not the study of its properties *per se*, but to take it as a representative of the class of equations satisfying condition (2),(3). Indeed for Eq. (4),

$$\omega(k) = \frac{1}{k + 1/k}. \quad (5)$$

Equation (4) represents in our case a toy model, making the problem tractable. By the derivations that follow, it is clear that we are not going to use any peculiar property of the BBM equation, besides the dispersion relation (5). In this sense, our results should hold approximatively even in cases where the dispersion relation is not known exactly, but the behavior given by Eqs. (2),(3) can be ascertained.

We note that our approach distinguishes itself from others which are found in the literature in the field: the theory of nonlinear resonant wave-wave interaction and the theory on nonlinear interaction between a long wave and a train of short waves. The first case is possible if the linear dispersion relation admits the three or the four-wave interaction process. This case has been investigated in Refs. [6–8]. The second case happens when the dispersion relation is such that

the phase velocity of the long-wave component coincides with the group velocity of the short wave (Benney's theory). The earliest studies along this line were made in Refs. [9–11] and later in Ref. [12].

II. BASIC PERTURBATIVE EXPANSIONS

Asymptotic analysis based on multiple-scale methods makes use of scaled variables, introducing order of magnitude relations between space, time, and amplitude variables. By fixing different relations, we define different perturbative expansions. In the present case, this goes as follows. To consider long space-scales we must introduce the variable $\xi = \epsilon x$, where $\epsilon \ll 1$, as is usual in long wavelength perturbative expansions [13]. To be able to consider short waves concomitantly a variable related to short-space scales is needed. In view of Eqs. (2),(3), the natural choice is $\zeta = \epsilon^{-1}x$, as has already been used in Ref. [14]. As for time variables, and in order to treat higher order effects [15], we introduce $\tau_1 = \epsilon t$, $\tau_3 = \epsilon^3 t$, $\tau_5 = \epsilon^5 t$, We now suppose that $u(x, t)$ is a function only of these scaled variables. This implies the following relations:

$$\partial_x = \epsilon \partial_\xi + \epsilon^{-1} \partial_\zeta, \quad (6)$$

$$\partial_t = \epsilon \partial_{\tau_1} + \epsilon^3 \partial_{\tau_3} + \epsilon^5 \partial_{\tau_5} + \dots \quad (7)$$

We will suppose that u can be written as an expansion of the form

$$u = \epsilon^N (u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \dots), \quad (8)$$

where N is a constant to be chosen. It fixes the relation between the longness/shortness parameters and the amplitude scale. It turns out that the interesting cases are $N=0,2$. Negative N would result in too strong coupling regime, and with a larger N nothing really new appears with respect to the previous cases. Let us now separately treat the cases $N=0,2$.

A. $N=0$

The leading order equation is a purely short-wave equation, first derived in Ref. [14], which reads

$$u_{0,\xi\tau_1} = u_0 - 3u_0^2. \quad (9)$$

Let us now take a particular solution to this equation, namely,

$$u_0 = \frac{1}{2} \operatorname{sech}^2 \left[k_s \zeta + \frac{1}{4k_s} \tau_1 + \phi \right]. \quad (10)$$

Here, k_s is a constant and ϕ is a phase which is allowed to depend on ξ , τ_3 , τ_5 , Expression (10) is a solution to the short-wave equation (9) but long-wave effects are allowed to appear in the phase. To determine these effects we must calculate the next orders and proceed to eliminate secular terms, a procedure that, and each order, fixes the dependence of ϕ on ξ and the higher order time variables. This is a straightforward procedure, whose details we will not address here. Calling Θ the argument of the sech^2 in Eq. (10), we obtain that

$$\Theta = k_s \zeta + k_l \xi + \omega_1 \tau_1 + \omega_3 \tau_3 + \Phi, \quad (11)$$

with $\omega_1 = 1/4k_s$, $\omega_3 = 1/16k_s^3 - k_l/4k_s^2$ and Φ a new phase depending only on τ_5 and higher-order time variables. The appearance of a term ω_3 coming from secularity elimination corresponds to renormalization of the wave speed. The term $k_l \xi$ introduces a redefinition of the wave number due to long-wave effects.

The calculation may be continued to next orders. This gives further corrections to Θ :

$$\Theta = k_s \zeta + k_l \xi + \omega_1 \tau_1 + \omega_3 \tau_3 + \omega_5 \tau_5 + \bar{\Phi}, \quad (12)$$

with $\omega_5 = 1/64k_s^5 + k_l^2/4k_s^3 - 3(k_l/16k_s^4)$.

B. $N=2$

The $N=2$ case corresponds to a weaker nonlinearity. Indeed the resulting perturbative expansion is made of linear equations. Nonetheless, the effects are worth comparing with the $N=0$ case. The lowest order equation is

$$u_{0,\xi} = u_{0,\xi\tau_1}, \quad (13)$$

which may be integrated once in ζ , yielding

$$u_{0,\xi\tau_1} = u_0, \quad (14)$$

which is just the linear part of Eq. (9). Thus, as lowest order equation we have again a purely short-wave one. we take again a particular solution for the lowest order, in this case

$$u_0 = A e^{i\theta} + A^* e^{-i\theta}, \quad (15)$$

where $\theta = k_s \zeta - (1/k_s) \tau_1$. A is an amplitude which may depend on ξ , τ_3 , τ_5 , . . . , and where the long-wave effects to the lowest order are to appear. We will again proceed to compute the next order and perform a secularity elimination procedure. The results can be summarized as follows. A is to satisfy a secularity elimination condition of the form:

$$A_{\tau_3} - \frac{1}{k_s^2} A_\xi = \frac{i}{k_s^3} A. \quad (16)$$

If we introduce $A = B e^{(i/k_s^2)\tau_3}$, where B is allowed to depend on ξ , τ_3 , τ_5 , . . . , we obtain the following equation for B :

$$B_{\tau_3} - \frac{1}{k_s^2} B_\xi = 0, \quad (17)$$

which implies that $B = B(\eta, \tau_5, \dots)$, where $\eta = \xi + (1/k_s^2)\tau_3$. B is otherwise undetermined, and we will need to compute the next order in perturbative expansion. Notice that, by defining $\theta_3 = \theta + (1/k_s^3)\tau_3$, the expression for u_0 becomes

$$u_0 = B e^{i\theta_3} + B^* e^{-i\theta_3}. \quad (18)$$

The same velocity renormalization effect as in the $N=0$ is present here. The computation now follows straightforwardly. The next-order secularity elimination condition is

$$B_{\tau_5} = i \left[\frac{1}{k_s^3} B_{\eta\eta} - \frac{1}{k_s^5} B - \frac{6}{k_s} B|B|^2 \right] - \frac{3}{k_s^4} B_{\eta}. \quad (19)$$

Let us now study some solutions to this equation, in order to obtain the effects of the long-space spaces, contained in η . First let us put

$$B = F(\eta, \tau_5) e^{i\Omega}, \quad (20)$$

where $\Omega = k_l \eta + \lambda \tau_5$, F is real, λ is a constant, still to be determined, depending on k_s and k_l . Equation (20) implies two equations, coming from the real and imaginary parts of Eq. (19):

$$F_{\tau_5} = - \left(\frac{2k_l}{k_s^3} + \frac{3}{k_s^4} \right) F_{\eta}, \quad (21)$$

$$\lambda F = \frac{1}{k_s^3} (F_{\eta\eta} - k_l^2 F) - \frac{1}{k_s^5} F - \frac{6}{k_s} F^3 - \frac{3k_l}{k_s^4} F. \quad (22)$$

A particular solution to this system can be verified to be

$$F = \frac{k_l}{\sqrt{3}k_s} \tanh(k_l \eta + V \tau_5), \quad (23)$$

$$\lambda = - \left(\frac{1}{k_s^5} + \frac{3k_l}{k_s^4} + \frac{3k_l^2}{k_s^3} \right), \quad (24)$$

where $V = -(2k_l/k_s^3 + 3/k_s^4)$. Summing up all of this, we write the expression for u_0

$$u_0 = \frac{k_l}{\sqrt{3}k_s} \tanh \left[k_l \left(\xi + \frac{1}{k_s^2} \tau_3 \right) + V \tau_5 \right] e^{i\theta_5} + \text{c.c.}, \quad (25)$$

where

$$\theta_5 = k_s \zeta + k_l \xi - \frac{1}{k_s} \tau_1 + \left(\frac{1}{k_s^3} + \frac{k_l}{k_s^2} \right) \tau_3 - \left(\frac{1}{k_s^5} + \frac{3k_l}{k_s^4} + \frac{3k_l^2}{k_s^3} \right). \quad (26)$$

This result represents the following: u_0 must be a solution of the short-wave equation Eq. (13), but long-wave effects reappear when secular terms are eliminated in the perturbative expansion. These long-wave effects represent a redefinition of the wave numbers, a renormalization of the velocity and, finally, a modulation of the amplitude depending of slow space and time scales.

III. FINAL REMARKS

We have considered in this article a (1+1)-dimensional system described by equations admitting a dispersion relation displaying the symmetry $k \rightarrow 1/k$. Thus, from a linear point of view, both short and long waves are well behaved. Considering the full nonlinear equations and admitting the fields to depend concomitantly on long-wave and short-wave variables, we have constructed a perturbative expansion valid over long times. The lowest order equation depends only on short-wave space variables (ζ), and the dependence on the long-space scale (ξ) is fixed only by considering the higher order effects in a properly constructed perturbative series, where secular terms are eliminated order by order. The results can be summarized as follows: if the typical amplitude is of $\mathcal{O}(1)$ there exists a solitary wave depending simultaneously on ξ and ζ , whose velocity is renormalized order by order. On the other hand, if the typical amplitude is small, of $\mathcal{O}(\epsilon^2)$, then the lowest order equation is linear, and again independent of ξ . With secular effects taken into account, the picture emerges of a periodic wave whose phase depends linearly on ξ and ζ , with a order-by-order renormalized velocity and with a modulated amplitude, effective over long-space scales.

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